Mortality of Iterated Piecewise Affine Functions over the Integers: Decidability and Complexity

Amir M. Ben-Amram

The Academic College of Tel-Aviv Yaffo

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Background: Dynamical systems

Discrete-time dynamical system

a state space X and an evolution function $f : X \to X$.

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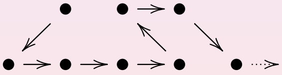

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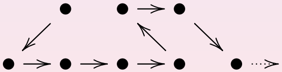

In Dynamical System theory, we are often interested in *asymptotic* properties of *all trajectories*.

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Halting problems are undecidable in general

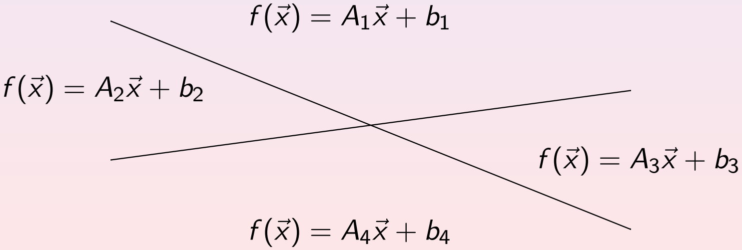
Halting, Termination, Mortality

- The halting problem considers halting of a given program with given input.
  - The halting problem is undecidable — but RE (Σ^0_1)
- Termination (in program verification) is called Totality in Recursion Theory.
  - the program should halt when initialized with any input
  - harder than the halting problem: Π^0_2-complete
- Mortality is more stringent: A computation started at *any state* should halt.
  - Herman (1969): Turing Machine mortality is undecidable.
  - Kurtz and Simon (2007): Counter Machine mortality is Π^0_2-complete.

Mortality for Simple Functions

Some simple classes of functions have been studied both in Dynamical System Theory and in Program Verification.

- Affine functions: $f(\vec{x}) = A\vec{x} + b$
- Piecewise affine functions. The space is divided (by hyperplanes) into a finite number of regions, and f is affine on each



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[BBKPT 2001], Theorem 2

Mortality is undecidable for piecewise-affine functions (with rational coefficients) over $\mathbb{R}^n$ for all $n \geq 2$.

[BBKPT 2001], Corollary 1

Mortality is decidable for continuous piecewise-affine functions in one dimension.

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Reduction from halting

"Reductions usually encode machine states as points in the space. Not all points represent states, which makes a reduction to mortality difficult."

Meanwhile, in Program Termination...

Very similar questions have been asked (and partly answered)

- (Un)Decidability results:
  - Tiwari — CAV 2004
  - Braverman — CAV 2006
  - Ben-Amram, Genaim, Masud — VMCAI 2011
- Heuristic solutions (many)

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Goal of this work

Characterize decidability of mortality for functions over $\mathbb{Z}^n$ (in particular, in terms of dimension).

Results

Consider piecewise-affine functions over $\mathbb{Z}^n$.

- Mortality is undecidable for $n = 2$.

- Mortality is decidable for $n = 1$.

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Consider piecewise-affine functions over $\mathbb{Z}^n$.

- Mortality is Π_2^0 complete for $n = 2$.
  - Also when the number of regions is bounded by some constant.
  - Also when the affine functions are monic:

    $f(x, y) = (x + a, y + b)$ or $f(x, y) = (y + b, x + a)$

    No multiplication involved
- Mortality is PSPACE-complete for $n = 1$.

Methods (undecidability proofs)

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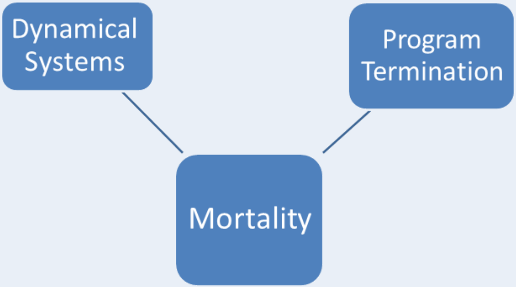

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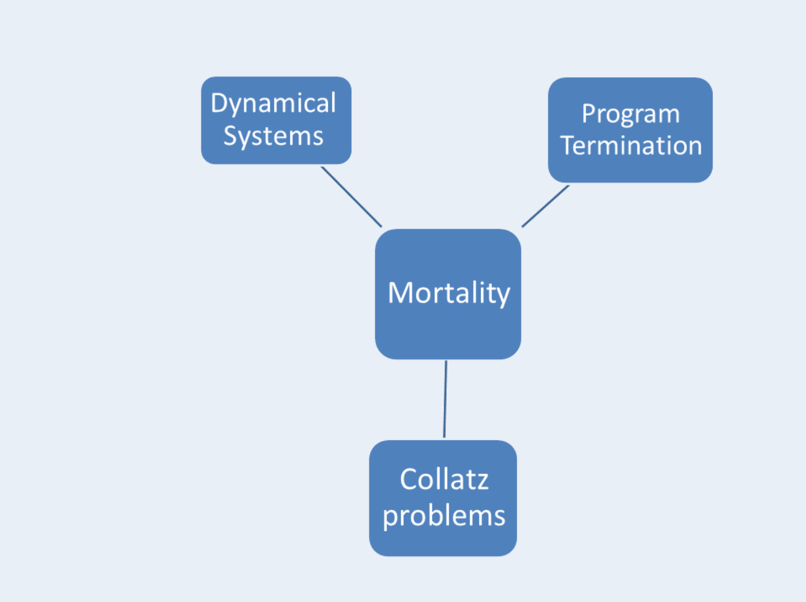

The Collatz Problem

Consider the function over $\mathbb{N}$:

$$g(x) = \begin{cases} 3x+1 & \text{if } x \bmod 2 = 1 \\ x/2 & \text{if } x \bmod 2 = 0 \end{cases}$$

Problem (posed by Lothar Collatz)

Do all trajectories of this function (over $\mathbb{N}$) converge to 1 ?

Example: $5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

- Celebrated open problem

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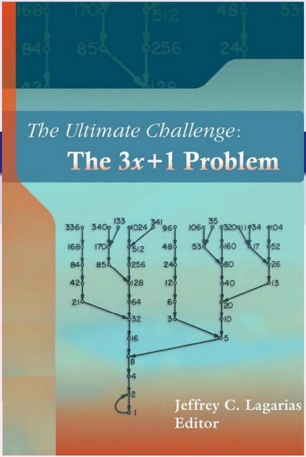

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Generalized Collatz Problems

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Definition

A generalized Collatz function has the form

$$g(x) = \begin{cases} a_0 y + b_0 & \text{if } x = my + 0 \\ a_1 y + b_1 & \text{if } x = my + 1 \\ \vdots & \\ a_{m-1} y + b_{m-1} & \text{if } x = my + (m-1) \end{cases}$$

For some *modulus* $m > 0$ and $a_0, b_0, \ldots, a_{m-1}, b_{m-1} \geq 0$.

GCP (Generalized Collatz Problem) is the problem of deciding whether every trajectory of g reaches 1.

Theorem (Kurtz & Simon 2007, extending Conway)

GCP is Π^0_2*-complete.*

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Theorem (new)

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Proof: reduction from the GCP.

Computing $g(x)$ involves: (1) division by m, (2) application of the function $f_i(y) = a_i y + b_i$ according to the remainder i.

The division is simulated by a trajectory in two dimensions.

Simulating a Collatz problem in $\mathbb{Z}^2$

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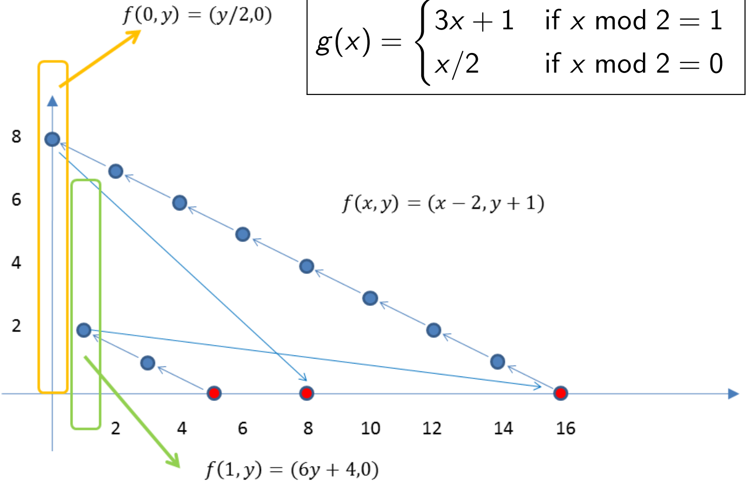


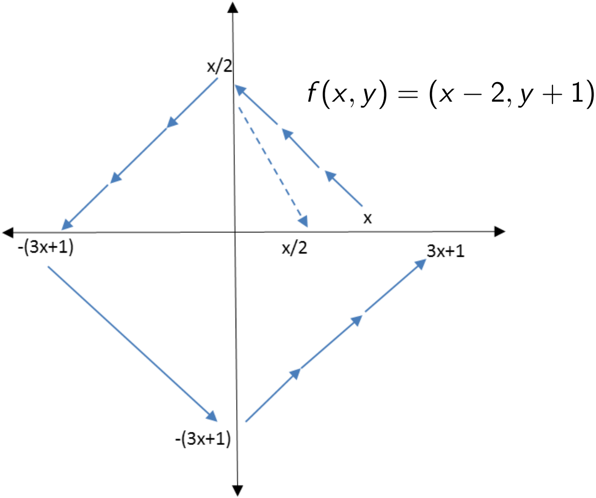
Monic Piecewise-Affine Functions

Recall that a monic function has the form
$f(x, y) = (x + a, y + b)$ or $f(x, y) = (y + b, x + a)$.

This prevents us from directly applying the function
$f_i(y) = a_i y + b_i$.

We have to simulate multiplication as well.

Simulating the $3x + 1$ function



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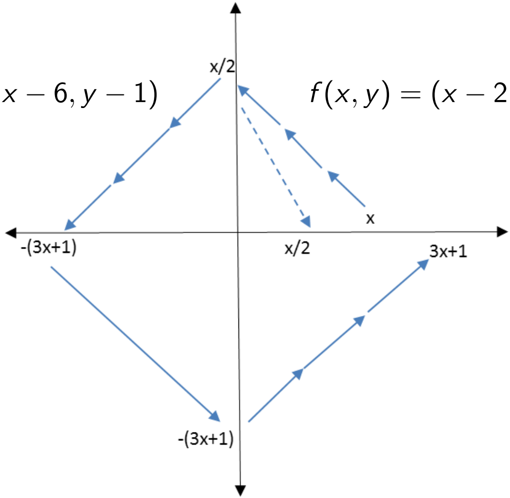
$f(x, y) = (x - 6, y - 1)$



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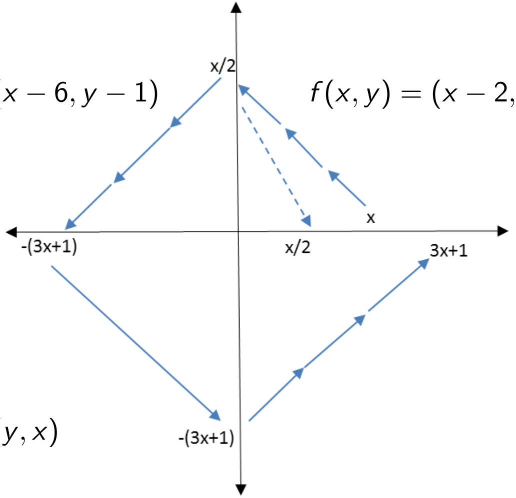

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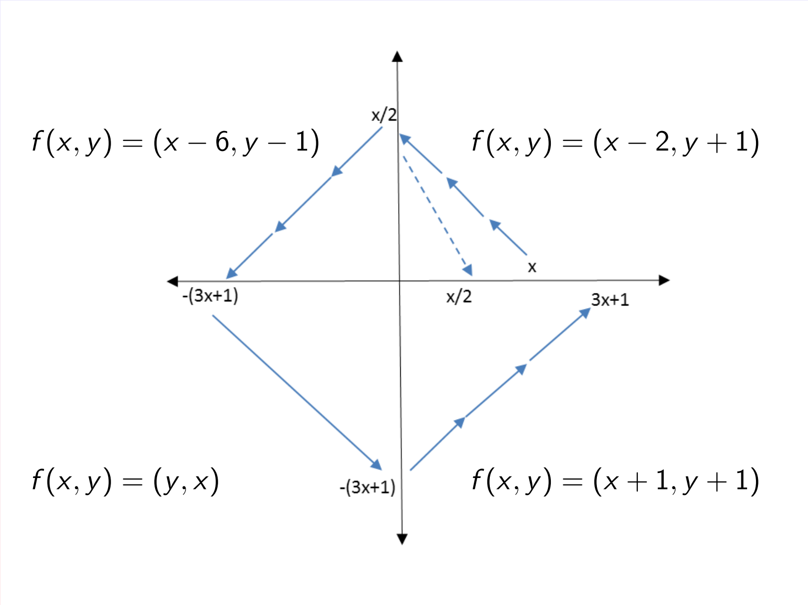

Simulating general Collatz functions?

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- I see no (direct) way to do it
- Instead, we introduce the Compass Collatz Functions

Compass Collatz Functions

Let $\mathcal{C} = \{E, N, W, S\}$.
The set of compass points is $\mathcal{P} = \mathbb{N} \times \mathcal{C}$.
A *Compass Collatz function* is $g : \mathcal{P} \to \mathcal{P}$ such that:

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The set of compass points is $\mathcal{P} = \mathbb{N} \times \mathcal{C}$.
A *Compass Collatz function* is $g : \mathcal{P} \to \mathcal{P}$ such that:
there is a number $m = 6p$ with $p \geq 5$ a prime, sets $R_N, R_S \subseteq [0, m-1]$ and integers $w_i \in [0, m-1]$, so that if $x = mx + rp + i$, where $0 \leq r < 6$, $0 \leq i < p$:

$$g(mx + rp + i, E) = \begin{cases} (mx + rp + i, N) & rp + i \in R_N \\ (4(mx + rp) + i, N) & rp + i \notin R_N \end{cases}$$

$$g(mx + rp + i, N) = (\frac{1}{2}mx + \lfloor \frac{1}{2}r \rfloor + i, W)$$

$$g(mx + rp + i, W) = \begin{cases} (mx + rp + w_{rp+i}, S) & rp + i \in R_S \\ (9(mx + rp) + w_{rp+i}, S) & rp + i \notin R_S \end{cases}$$

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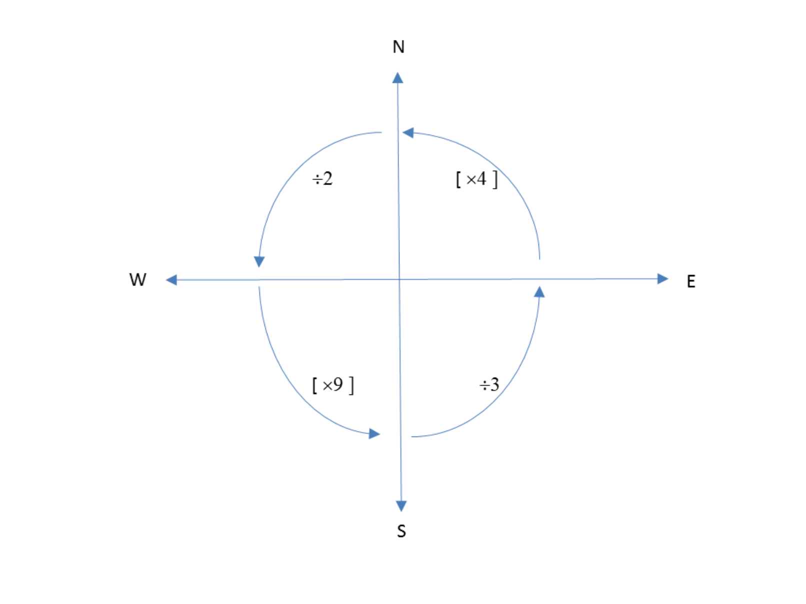

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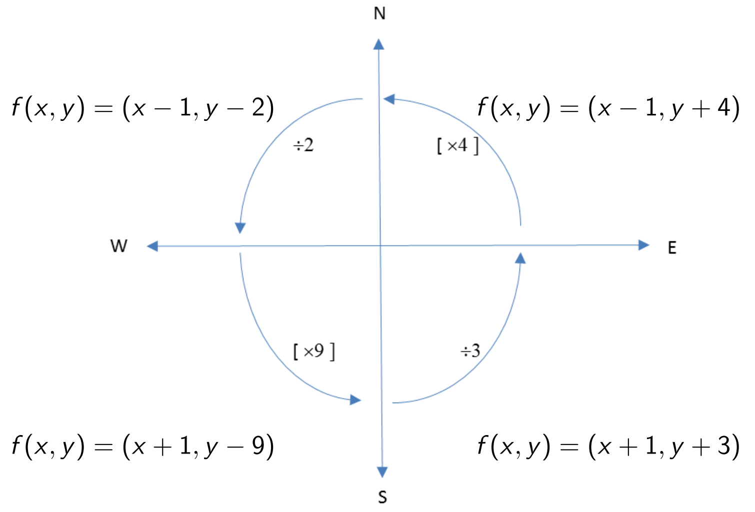

Concluding the proof

- A mortality problem for Compass Collatz-like Functions is shown Π^0_2-complete, by reduction from mortality of 2-counter machines.
- This problem is reduced to mortality of monic piecewise-affine functions on $\mathbb{Z}^2$.

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- The number of regions in the dynamical system is related to the size of the counter machine.
- For any constant c, mortality of counter machines smaller than c is a decidable problem!
- The undecidability proof uses a sort of enhanced counter machine.
- This technique does not extend to monic functions.

Loops with a convex guard

In Program Termination analysis, loops are often of the form

$$
\begin{array}{l}
\texttt{while } x \in G \texttt{ do} \\
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For what parameters is the problem decidable?